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LINEARIZATION BY NONLOCAL TRANSFORMATIONS AND ISOCHRONICITY CONDITION OF THE LIÉNARD EQUATIONS

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Abstract. We consider the linearization problem for the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ using nonlocal transformation and its hamiltonization by means of Jacobi's Last multiplier. In this context we find that when the equation is mapped to that of the linear harmonic oscillator by means of a nonlocal transformation then there appears, almost naturally, a relation between the functions $f(x)$ and $g(x)$ which is exactly similar to that derived by M. Sabatini in (J. Diff. Eqns. 152,467-487, 1999) in the context of isochronicity of the Liénard equation. Furthermore, a complex Hamiltonian formulation of the Liénard system is also discussed.

Keywords. Nonlocal transformation, linearization, isochronous, Liénard system. coupled equation.

1 Introduction

The linearization of a nonlinear ordinary differential equation (ODE) has been an object of immense interest for many years. The most commonly employed method is to seek a point transformation such that the transformed ODE is linear and hence may be solved by some known method.

Of late, however, a number of attempts have been made to tackle this problem by using Sundman transformations [6], which are nonlocal in character. Besides, second-order ODEs, for which the linearization problem is well studied, Euler *et al* have also extended their procedure to deal with higher-order (mainly third-order) ODEs. The most general type of a nonlocal transformation that may be considered is of the form

$$dX = A(x,t)dx + B(x,t)dt, \quad dT = C(x,t)dx + D(x,t)dt. \tag{1}$$

The usual case of a point transformation corresponds to the situation where $A_t(x,t) = B_x(x,t)$ and $C_t(x,t) = D_x(x,t)$ so that $X(x,t) = F(x,t)$ and $T(x,t) = G(x,t)$. In [6] as also in [5] it was assumed that $X(x,t) = F(x,t)$ but that $C_t(x,t) \neq D_x(x,t)$, so that the temporal part is nonlocal. In fact they took $C(x,t) = 0$ so that $dT = D(x,t)dt$ assuming $D_x \neq 0$. Such a nonlocal transformation is commonly referred to as a Sundman transformation.

In this paper we have attempted to first of all generalize this formalism with suitable modification to identify, at least partially, certain novel features related to the isochronicity of a well known planar dynamical system, namely the Liénard differential equation.

A potential $V : \mathbb{R} \to \mathbb{R}$ is said to be isochronous if all periodic solutions of $\ddot{x} = -V'(x)$ have the same period. Harmonic oscillators are the simplest examples but there are many others.

The Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{2}$$

has been extensively studied owing to its diverse physical applications and its appearance in the context of limit cycles of the Van der Pol equation. Here we dwell on the aspect of isochronicity of such an equation. In [17] the author has studied the monotonicity properties of the period function of (2). In particular it is shown that if the functions f and g be analytic, g odd, $f(0) = g(0) = 0$ and $g'(0) > 0$ then the origin is an isochronous center if and only if f is odd and

$$\tau(x) := \left(\int_0^x sf(s)ds\right)^2 - x^3(g(x) - g'(0)x) \equiv 0. \tag{3}$$

Here we obtain the condition stated in (3) using a nonlocal transformation. It turns out that nonlocal transformations apparently play an important role in the theory of isochronous systems. An interesting example of the Liénard equation with isochronous solutions has been given by Chandrasekar et al. [3], who have shown that, when $\lambda > 0$,

$$\ddot{x} + 3k\dot{x}x + \lambda x + k^2x^3 = 0 \tag{4}$$

has the explicit solution

$$x(t) = \frac{A\sin(\omega t + \mu)}{1(k/\omega)A\cos(\omega t + \mu)},$$

with $\omega = \sqrt{\lambda}$, and μ an arbitrary constant. For $0 < A < \omega/k$, this yields isochronous oscillations of frequency ω,

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